MTH 508/608: Introduction to Differentiable Manifolds and Lie Groups
Assignment 2: Vector fields on manifolds

1 Practice problems

1. Show that the relation $\sim$ in Definition 2.1 (i) of the Lesson Plan is indeed an equivalence relation and that $\mathcal{C}^\infty(p)$ forms an algebra over $\mathbb{R}$.

2. Establish (or study the proofs of) the following assertions in the Lesson Plan: Theorem 2.1 (iv)(a), 2.1 (xii)(a), Theorem 2.3 (xii), Theorem 2.3 (xiii), Lemma 2.4 (xi), Corollary 2.4 (xii), Theorem 2.5 (iv), Corollary 2.5 (v), Corollary 2.6 (v), Example 2.6 (vi), Theorem 2.6 (viii), Theorem 2.7 (x), Corollary 2.7 (xii), Theorem 2.7 (xiv), Corollary 2.8 (vii), Theorem 2.8 (viii), Theorem 2.9 (iii), Lemma 2.9 (iv), and Theorem 2.9 (v).

3. Read carefully through Example 1.10 worked out in pages 110-111 of Boothby.

   (a) Using the notation in this example, show that for any $\alpha, \beta \in \mathbb{R}$, there exists a parametrized curve on M through p whose velocity vector is exactly $\alpha(X_u)_0 + \beta(X_v)_0$.

   (b) Let the surface in the example be parametrized in the form $z = h(x, y)$ with $z_0 = h(x_0, y_0)$. Show that under suitable parametrization, the tangent plane $T_{(x_0,y_0,z_0)}(M)$ consists of all vectors from (x_0, y_0, z_0) to (x, y, z) satisfying
   $$\left(\frac{\partial h}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial h}{\partial y}\right)_0 (y - y_0) - (z - z_0) = 0.$$

4. Let M be a smooth n-manifold and let $T(M) = \cup_{p\in m} T_p(M)$ be the *tangent bundle* of M. There is a natural *projection map* $\pi : T(M) \to M$ that sends each vector in $T_p(M)$ to the point that which it is tangent.

(a) For any coordinate neighborhood (U, φ) of M, define

$$\tilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$$

by

$$\tilde{\varphi}\left(v_i\left(\frac{\partial}{\partial x_i}\right)_p\right) = (x_1(p), \ldots, x_n(p), v_1, \ldots, v_n),$$

where the x_i are the coordinate functions of φ. Using $\tilde{\varphi}$ establish that TM has a smooth structure that makes it a $2n$ dimensional smooth manifold.

(b) If $F : N \to M$ is C^∞, show that $F_* : T(N) \to T(M)$ is C^∞ and $\pi \circ F_* = F \circ \pi$.

5. Let M be a smooth manifold. Show the following statements are equivalent.

   (a) X is a C^∞ vector field on M.

   (b) Whenever f is a C^∞ function in a open set $W_f \subset M$, Xf defined by $(Xf)_p = X_p f$ is C^∞ on W_f.

   (c) X is a continuous map $X : M \to TM$ such that $\pi \circ X = id_M$.

6. Show that if $F : N \to M$ is smooth and X be a C^∞ vector field on N, then $Y = F_*(X)$, if it exists, is uniquely determined if and only if $F(N)$ is dense in M.

7. Let : $\tilde{M} \to M$ be smooth covering and Y a smooth vector field on M. Show that there exists a unique smooth vector field X on $\tilde{M}$ such that $F_*(X) = Y$.

8. Determine the infinitesimal generator X^θ for the following actions θ.

   (a) $\theta : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by

   $$\theta(t, (x, y)) = (-x\cos(t) + y\sin(t), -x\sin(t) + y\cos(t)).$$

   (b) $\theta : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\theta(t, (x, y)) = (-xe^{2t}, ye^{-3t})$.

   (c) $\theta : \mathbb{R} \times \mathrm{GL}(2, \mathbb{R}) \to \mathrm{GL}(2, \mathbb{R})$ defined by

   $$\theta(t, A) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot A.$$

9. Determine the one-parameter action θ^X associated with the following actions vector fields X.

   (a) The vector field $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ on $\mathbb{R}^2$.

(b) The vector field $X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ on $\mathbb{R}^2$.

(c) The vector field $X = x^2\frac{\partial}{\partial x}$ on $\mathbb{R}$.

10. Let X and Y be smooth vector fields on N and M, respectively, and let $F : N \to M$ be a smooth mapping. Then show that $F_*(X) = Y$ if and only if $F \circ \theta_t^X(p) = \theta_t^Y \circ F(p)$ for all (t, p) where both sides are well-defined.

11. Given $p \in M$, if $(\alpha(p), \beta(p))$ is bounded for a C^∞ vector field X on M, then show that $t \to \theta(t, p)$ is an imbedding.

12. Show that a non-trivial one-parameter subgroup H of a Lie group G is either an isomorphic image of S^1 or $\mathbb{R}$.

13. Show that for a Lie group G, the map $\mu : G \to G$ defined by $\mu(g) = g^{-1}$, takes left-invariant vector fields to right-invariant vector fields.

14. Consider $A \in \mathrm{GL}(n, \mathbb{R})$ and $X \in M_n(\mathbb{R})$.

(a) Show that $Ae^X A^{-1} = e^{AXA^{-1}}$.

(b) Using (a), show that $\det(e^X) = e^{\mathrm{tr}(X)}$

(c) Determine all matrices A such that $\{e^{tA} : t \in \mathbb{R}\}$ is a one-parameter subgroup of $\mathrm{SL}(n, \mathbb{R})$.

15. Consider the set $\mathfrak{X}(M)$ of C^∞ vector fields on M.

(a) Show that $\mathfrak{X}(M)$ is an infinite-dimensional vector space over $\mathbb{R}$

(b) Show that $\mathfrak{X}(M)$ is a module over $C^\infty(M)$.

(c) Show that $\mathfrak{X}(M)$ locally finitely generated over $C^\infty(M)$, that is, each $p \in M$ has a neighborhood V on which there is a finite set of vector fields that generate $\mathfrak{X}(M)$ as a $C^\infty(V)$ module.

16. Show that for $X, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$, we have

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X.$$

17. For $G = \mathrm{GL}(n, \mathbb{R})$, consider the Lie algebra $\mathfrak{g}$. Consider the map $\mu : \mathfrak{g} \to M_n(\mathbb{R})$ that assigns to each $X \in \mathfrak{g}$ the matrix $A = (a_{ij})$ of components of X_e. Show that μ is an algebra isomorphism, that is,

$$\mu[X, Y] = \mu(X)\mu(Y) - \mu(Y)\mu(X).$$

18. If $F : M \to N$ is a diffeomorphism and $X, Y \in \mathfrak{X}(M)$, then show that

$$F_*(L_X Y) = L_{F_*(X)} F_*(Y).$$

19. If $f \in C^\infty(M)$, $X, Y \in \mathfrak{X}(M)$, and $L_X f = Xf$, then show that

$$L_X(fY) = (L_X f)Y + f(L_X Y).$$

20. A vector field X on M is said to *belong* to a distribution Δ on M if for each $p \in M$, we have $X_p \in \Delta_p$. Show that a C^∞ distribution Δ on M is involutive if and only if for every pair of C^∞ vector fields X, Y on M that belong to Δ, we have $[X, Y]$ belongs to Δ.

21. Let N be a maximal integral manifold of a distribution Δ on M. Show that if N is closed, then N is a regular submanifold of M.

22. Let $N \subset M$ be a submanifold, and let $X, Y \in \mathfrak{X}(M)$ be such that if $p \in N$, then $X_, Y_p \in T_p(N)$. Then show that $p \in N$ implies that $[X, Y]_p \in T_p(N)$.

23. Let G be a Lie group and H a closed Lie subgroup such that $H \lhd G$. Then show that G/H is Lie group and $G \to G/H$ is a Lie group homomorphism.

24. Let $F : G_1 \to G_2$ be a Lie group homomorphism. Show that ker F is a closed Lie subgroup of G_1.

25. Show that every countable subset of $\mathbb{R}^k$ has an isolated point.

26. Let H be a Lie group and let H_0 be the component of e.
    (a) Show that H has at most a countable countable number of connected components that are all open and closed and diffeomorphic to H_0.
    (b) Show that $H_0 \lhd H$ and that H/H_0 is a discrete Lie group.

2 Problems for submission

- **Homework 3 (Due 30/10/24)**: Establish the assertions marked in red in the Midterm solutions.
- **Homework 4 (Due 14/11/24)**: Solve problems 3, 5, 12, 17, and 26 from the practice problems above.